THEORY OF ELASTICITY WITH SPATIAL DISPERSION. ONE-DIMENSIONAL COMPLEX STRUCTURE

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An elastic medium of simple structure with spatial dispersion was considered in [1]. In the present paper we construct a more general model of a macroscopically homogeneous medium of complex structure which cannot be described adequately in terms of a single kinematic variable. As our initial micromodel we have chosen the familiar model of a complex chain each of whose unit cells has N degrees of freedom [2 and 3]. In sec. 1 this model is generalized for the case of a continuous mass distribution. New kinematic variables, namely the displacement of the centers of mass of the cells and microdeformations of various orders, are introduced. Appropriately, force micromoments are introduced as the force variables. The algorithm introduced in [1] is used to effect a transition to equations of an elastic medium with spatial dispersion. The corresponding operations are expressed explicitly in terms of the initial microparameters.

With the phenomenological approach, the resulting equations describe the most general one-dimensional model of a homogeneous linearly elastic medium of complex structure with spatial dispersion. We also consider the general equations of a nonhomogeneous medium of periodic structure with a single kinematic variable. It is established that under certain conditions both of these models can be considered as representations of one and the same physical model, but with different areas of applicability. Formulas for converting from one representation to the other are presented.

Section 2 contains a discussion of certain specific models and a derivation of the sufficient conditions under which the equations of a medium of complex structure admit of exact transformation into the equations of a medium of simple structure. It is shown that in the case of weak dispersion, the equations of a medium of complex structure with certain additional limitations coincide with the one-dimensional equations of the couple stress theory of elasticity [4 to 6]. It is difficult to justify these limitations physically, however.

The acoustic frequency range is of primary interest in elasticity theory. In sec. 3 it is shown that in the acoustic range it is always possible to transform the system of equations describing a complex structure into an equation with a single kinematic variable (the displacement of the centers of mass of the cells) and into equations explicitly solved

for the remaining variables. This involves both spatial and temporal dispersion, although the latter is not related to energy dissipation.

It is important to note that the operators which appear in the equation involving the centers of mass are directly related to macroexperiment. Specifically, in the zeroth approximation there occurs a transition to the ordinary equation of elasticity theory with an elasticity constant which is determined by experiment. Because of this, it is our view that this representation is more adequate to the macroscopic description of a medium of complex structure than the representations considered in sec. 1.

1. We shall proceed on the basis of the Born model of an unbounded one-dimensional complex chain in the harmonic approximation [2 and 3]. Each unit cell numbered n contains N particles with masses m_j, each particle interacting elastically both with the particles of its own cell and with the particles of the other cells. The corresponding Lagrangian is of the form

$$2L = \sum_{nj} m_j \dot{w}(n, j)\, \dot{w}(n, j) - \\ - \sum_{nn'jj'} w(n, j)\, \Phi(n - n', j, j')\, w(n', j') + 2\sum_{nj} w(n, j)\, f(n, j) \qquad (1.1)$$
$$(j, j' = 1, \ldots, N)$$

Here $w(n, j)$ is the displacement of the j-th particle in the n-th cell, $f(n, j)$ is the external force acting on it, and $\Phi(n - n', j, j') = \Phi(n' - n, j', j)$ are the force constants (the characteristics of the corresponding springs in the mechanical analog). The dependence of the force constants on the difference $n - n'$ is a consequence of the fact that the structure of the unit cell recurs periodically in the chain. The requirement of the invariance of energy with respect to translation imposes certain conditions on the force constants [2]. We note that because of considerations similar to those cited in [1], the term linear with respect to displacement is omitted in the Lagrangian, i.e. initial forces are assumed to be lacking.

Let us generalize this model for the case of an arbitrary (discrete and continuous) periodic mass distribution. We introduce the local cell coordinate ξ with its origin at the center of mass of the cell and denote by $\rho(\xi)$ the density of masses in the cell. The Lagrangian then becomes

$$2L = \sum_n \int \rho(\xi)\, \dot{w}(n, \xi)\, \dot{w}(n, \xi)\, d\xi - \\ - \sum_{nn'} \iint w(n, \xi)\, \Phi(n - n', \xi, \xi')\, w(n', \xi')\, d\xi\, d\xi' + 2\sum_n \int w(n, \xi)\, f(n, \xi)\, d\xi \qquad (1.2)$$

To make the transition to a discrete mass distribution it is sufficient to set

$$\rho(\xi) = \sum_j m_j \delta(\xi - \xi_j), \qquad \Phi(n, \xi, \xi') = \sum_{jj'} \Phi(n, j, j')\, \delta(\xi - \xi_j)\, \delta(\xi' - \xi_{j'}) \qquad (1.3)$$

where ξ_j is the coordinate of the particle in the cell.

In order to avoid discussion of the convergence of the corresponding sums in (1.2), we shall assume that the effective range is limited, i.e. that $\Phi(n, \xi, \xi')$ differ from zero

for only a finite number of values n. As regards $w(n, \xi)$, we assume sufficient smoothness with respect to ξ and an increase not more rapid than of degree n as $|n| \to \infty$.

Considering the coordinate ξ in (1.1) as a parameter and making use of the algorithm cited in [1], we pass from the function of the discrete argument n to functions of the continuous variable x. The Lagrangian L in the (k, ω)-representation* then becomes (a is the size of the cell)

$$4\pi aL = \omega^2 \int\int \rho(\xi)\, \overline{w(k, \xi)}\, w(k, \xi)\, dk\, d\xi -$$

$$- \int\int\int \overline{w(k, \xi)}\, \Phi(k, \xi, \xi')\, w(k, \xi')\, dk\, d\xi\, d\xi' + 2\int\int \overline{w(k, \xi)}\, f(k, \xi)\, dk\, d\xi \quad (1.4)$$

In making the transition to elasticity theory it is convenient to introduce collective cell variables. Let us define the moments of inertia of order q ($p = 0, 1, \ldots$),

$$\rho^{(q)} = \frac{1}{a}\int \rho(\xi)\, \xi^q\, d\xi, \quad \text{or} \quad \rho^{(q)} = (\rho, \xi^q) \quad (1.5)$$

where the parentheses denote the corresponding scalar product, and introduce the diagonal matrices

$$I^{qq'} = \rho^{(2q)}\delta^{qq'}, \qquad I^{-1}_{qq'} = \rho^{-1}_{(2q)}\delta_{qq'} \quad (1.6)$$

With the aid of the familiar algorithm, let us construct an orthonormal system of polynomials $e^q(\xi)$ with the weight $\rho(\xi)$ and the reciprocal system of functions $e_q{}'(\xi)$, defining them by the relations

$$(\rho e^q, e^{q'}) = I^{qq'}, \qquad (e_q, e^{q'}) = \delta^{q'}_q \quad (1.7)$$

It is easy to show that**

$$e_q(\xi) = \rho(\xi)\, I_{qq'}{}^{-1} e^{q'}(\xi) \quad (1.8)$$

The biorthogonal functional basis which we have constructed is fully defined by specifying $\rho(\xi)$. In particular, when density is constant, $e^q(\xi)$ coincide with the Legendre polynomials. In the case of a discrete mass distribution the basis automatically turns out to be finite: $q = 0, 1, \ldots, N-1$.

Independently of $\rho(\xi)$, for the first two elements of the basis we have

$$e^0(\xi) = 1, \quad e^1(\xi) = \xi, \quad e_0(\xi) = \rho_{(0)}{}^{-1}\rho(\xi), \quad e_1(\xi) = \rho_{(2)}{}^{-1}\rho(\xi)\,\xi \quad (1.9)$$

Now let us expand all of the functions ξ in terms of the basis elements,

$$w(k, \xi) = w_q(k)\, e^q(\xi), \qquad f(k, \xi) = f^q(k)\, e_q(\xi)$$
$$\Phi(k, \xi, \xi') = \Phi^{qq'}(k)\, e_q(\xi)\, e_{q'}(\xi') \quad (1.10)$$

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* We recall [1] that functions of k are here the Fourier transforms of functions of x, the point k belonging to the circle B of radius α^{-1}. Here and below the bar denotes the complex conjugate.

** Here and below, summation over recurrent subscripts and superscripts is assumed.

The coefficients of the expansion are defined in the obvious way,

$$w_q(k) = (e_q, w), \qquad f^q(k) = (e^q, f)$$
$$\Phi^{qq'}(k) = \frac{1}{a^2}\iint \Phi(k, \xi, \xi')\, e^q(\xi)\, e^{q'}(\xi')\, d\xi\, d\xi' = \overline{\Phi^{q'q}(k)} \tag{1.11}$$

The first coefficients in the expansion have a simple physical meaning. Taking account of (1.9), we find that w_0 is the displacement of the center of mass of the cell, w_1 is the average cell deformation (microdeformation), f^0 is the average force density, and f^1 is the average force dipole density. The remaining coefficients correspond to microdeformations and micromoments of higher orders.

In the new variables the Lagrangian (1.4) can be written as

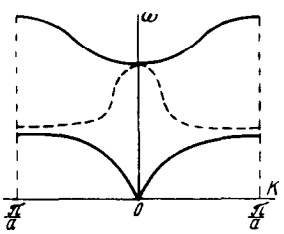
$$2L = \langle w_q|\omega^2 I^{qq'}|w_{q'}\rangle - \langle w_q|\Phi^{qq'}|w_{q'}\rangle + 2\langle w_q|f^q\rangle \tag{1.12}$$

in terms of the symbols employed in [1].

The corresponding equations are of the form

$$\omega^2 I^{qq'} w_{q'}(k) - \Phi^{qq'}(k)\, w_{q'}(k) = -f^q(k) \tag{1.13}$$

As we know [2 and 3], the free oscillations of the complex chain are described in the (k, ω)-representation by a single acoustic branch (extending from the origin) and $N - 1$ optical branches. In the figure these are schematically shown as solid curves for a diatomic chain. On transition to a continuous mass distribution, the number of optical branches increases without limit. The corresponding transmission bands in the plane k, ω will be called the acoustic and optical ranges.



Of paramount interest in elasticity theory is the acoustic range, in which the principal kinematic variable is the displacement of the cell center of mass. Because of this, it is advisable to isolate this variable.

Let us introduce the symbols ($p = 1, 2, \ldots$)

$$u = w_0, \quad \eta_p = w_p; \quad q = f^0, \quad \mu^p = f^p \tag{1.14}$$

In accordance with the foregoing, we break down the matrix $\Phi^{qq'}$ into blocks. The condition of energy invariance with respect to translation implies that $\Phi^{pp'}(k)$ can be represented in the form ($p, p' = 1, 2, \ldots$)

$$\Phi(k) = \begin{pmatrix} k^2\gamma(k) & ik\chi^{p'}(k) \\ \cdots\cdots & \cdots\cdots \\ -ik\chi^{p}(k) & \Gamma^{pp'}(k) \end{pmatrix} \tag{1.15}$$

where $\gamma(k)$, $\chi^p(k)$ and $\Gamma^{pp'}(k) = \overline{\Gamma^{p'p}(k)}$ are complete analytic functions uniquely defined by the specification of the initial matrix $\Phi(k)$.

Equations (1.13) can now be rewritten in the form of the system ($\rho = \rho_{(0)}$)

$$\begin{aligned} &\omega^2\rho\, u(k) - k^2\gamma(k)\, u(k) - ik\chi^p(k)\, \eta_q(k) = -q(k) \\ &\omega^2 I^{pp'}\eta_{p'}(k) + ik\chi^p(k)\, u(k) - \Gamma^{pp'}(k)\, \eta_{p'}(k) = -\mu^p(k) \end{aligned} \tag{1.16}$$

The transition to the (x, t)-representation is obvious: the functions γ, χ^p and $\Gamma^{pp'}$ are associated with integral operators with difference kernels.

The elastic energy Φ can be written as

$$\Phi = \int \varphi(x)\, dx \tag{1.17}$$

where $\varphi(x)$ is defined to within divergent terms. With the additional condition of invariance with respect to translation, $\varphi(x)$ is defined uniquely and can be interpreted as the density of the elastic energy. From (1.12), with allowance for (1.14) and (1.15), we find that

$$2\varphi(x) = \varepsilon(x)\,[\gamma(x) * \varepsilon(x)] + \eta_p(x)\,[2\chi^p(x) * \varepsilon(x) + \Gamma^{pp'}(x) * \eta_{p'}(x)] \tag{1.18}$$

Here $\varepsilon(x) = u'(x)$ is the macroscopic deformation, and the operation * denotes integral convolution.

With the condition of boundedness of the carriers $u(k)$ and $\eta_p(k)$, this model as described by Equations (1.16) is an exact representation of the initial model with a periodic discrete or continuous mass distribution. Here $\gamma(k)$, $\chi^p(k)$, and $\Gamma^{pp'}(k)$ are expressed explicity in terms of the microparameters of $\Phi(n, \xi, \xi')$. Conversely, for specified $\gamma(k)$, $\chi^p(k)$, and $\Gamma^{pp'}(k)$ which satisfy the obvious conditions it is possible to find $\Phi(n, \xi, \xi')$. With a phenomenological approach, system (1.16) can be considered as the most general one-dimensional model of a macroscopically homogeneous linearly elastic system of complex structure with spatial dispersion*.

It is important to note that a medium with a periodic structure can be described by a method different from that adopted in the Born model. In place of $w(n, \xi)$ and $f(n, \xi)$ let us introduce the displacement $w(x)$ and force density $f(x)$ which depend on the single spatial coordinate x. In these variables the Lagrangian can be written as

$$2L = \int \rho(x)\, w^{\cdot 2}(x)\, dx - \iint w(x)\, \Phi(x, x')\, w(x')\, dx dx' + 2\int w(x)\, f(x)\, dx \tag{1.19}$$

and the conditions

$$\rho(x) = \rho(x + a), \qquad \Phi(x, x') = \Phi(x + a, x' + a) \tag{1.20}$$

must be fulfilled.

Let us denote by $w(\varkappa)$, and $f(\varkappa)$ the Fourier transforms of $w(x)$ and $f(x)$, where $\varkappa$ (in contrast to $k \in B$) belongs to the number axis $R_\varkappa$. Further, let $\rho(\varkappa)$ be the Fourier inverse image of $\rho(x)$ and $\Phi(\varkappa, \varkappa')$ the Fourier transform of $\Phi(x, x')$ with respect to x and its Fourier inverse image with respect to x'. Then

$$\begin{aligned} 4\pi L = {} & \omega^2 \iint \overline{w(\varkappa)}\, \rho(\varkappa' - \varkappa)\, w(\varkappa')\, d\varkappa\, d\varkappa' - \\ & - \iint \overline{w(\varkappa)}\, \Phi(\varkappa, \varkappa')\, w(\varkappa')\, d\varkappa\, d\varkappa' + 2\int \overline{w(\varkappa)}\, f(\varkappa)\, d\varkappa \end{aligned} \tag{1.21}$$

In contrast to (1.13), the equation of motion can be written as the integral equation

* A medium of complex structure was considered from a different standpoint in [7].

$$\omega^2 \int \rho(\varkappa' - \varkappa) w(\varkappa')\, d\varkappa' - \int \Phi(\varkappa, \varkappa') w(\varkappa')\, d\varkappa' = -f(\varkappa) \tag{1.22}$$

It can be shown that under condition (1.20), expressions (1.13) and (1.22) are exactly equivalent representations of one and the same physical model. Let us cite without proof the explicit formulas for effecting the transition from one representation to the other. To do this we introduce the periodic δ-function

$$\delta_*(\varkappa) \equiv \frac{a}{2\pi} \sum_n e^{ina\varkappa} = \sum_n \delta\left(\varkappa - \frac{2\pi n}{a}\right) \tag{1.23}$$

The transition from (1.22) to (1.13) is given by the relations ($k \in B$, $\varkappa \in R_\varkappa$)

$$w_q(k) = \frac{1}{a}\int \delta_*(k - \varkappa)\, e_q(\varkappa)\, w(\varkappa)\, d\varkappa, \qquad f^q(k) = \frac{1}{a}\int \delta_*(k - \varkappa)\, \overline{e^q(\varkappa)}\, f(\varkappa) d\varkappa$$
$$\Phi^{qq'}(k) = \frac{1}{a^2}\iint \delta_*(k - \varkappa)\, e^q(\varkappa)\, \overline{e^{q'}(\varkappa')}\, \Phi(\varkappa, \varkappa')\, d\varkappa\, d\varkappa' \tag{1.24}$$

while the matrix $I^{qq'}$ is given by (1.7). The inverse formulas are

$$w(\varkappa) = \frac{1}{a} e^q(\varkappa) \int \delta_*(\varkappa - k) w_q(k)\, dk, \qquad f(\varkappa) = \frac{1}{a} e_q(\varkappa) \int \delta_*(\varkappa - k) f^q(k)\, dk$$
$$\rho(\varkappa - \varkappa') = \frac{1}{a^2} \delta_*(\varkappa - \varkappa') I^{qq'} e_q(\varkappa) \overline{e_{q'}(\varkappa')} \tag{1.25}$$
$$\Phi(\varkappa, \varkappa') = \frac{1}{a^2} \delta_*(\varkappa - \varkappa') e_q(\varkappa)\, \Phi_*^{qq'}(\varkappa)\, e_{q'}(\varkappa')$$

where $\Phi_*^{qq'}(\varkappa)$ is the periodic continuation of $\Phi^{qq'}(k)$. It is easy to show that conditions (1.20) are fulfilled here.

We also take note of the relations

$$\Phi_*^{qq'}(\varkappa) e_q(\varkappa) = \int \Phi(\varkappa, \varkappa') e^{q'}(\varkappa')\, d\varkappa', \qquad I^{qq'} e^q(\varkappa) = \int \rho(\varkappa' - \varkappa) e^{q'}(\varkappa')\, d\varkappa' \tag{1.26}$$

Thus, the above formulas establish the identity of the models of a macroscopically homogeneous medium with spatial dispersion as described by Equations (1.16) and that of a medium of periodic structure as described by Equation (1.22) under conditions (1.20). Nonetheless, the first model is, in a number of cases, more convenient. This is the case, for example, in considering wave processes and in converting to long-wave approximations, i.e. in the macrodescription of the medium.

2. Let us consider some of the more important special models. Let the unit cell have a center of symmetry. It can be shown that in this case

$$e^q(-\varkappa) = (-1)^q e^q(\varkappa), \qquad \Phi(\varkappa, \varkappa') = \Phi(\varkappa', \varkappa)$$

From (1.24) it follows immediately that

$$\Phi^{qq'}(-k) = \overline{\Phi^{qq'}(k)} = (-1)^{q+q'} \Phi^{qq'}(k) \tag{2.1}$$

At the same time, it is difficult to indicate any real model in the case of the stronger condition $\Phi^{qq'}(-k) = \Phi^{qq'}(k)$.

It is of interest to determine the conditions under which Equation (1.16) of a medium of complex structure admits of transformation into the equations of a medium of simple structure [1]. Clearly, in this case $\Phi(x, x') = \Phi(x - x')$ so that $\Phi(\varkappa, \varkappa') = \Phi(\varkappa)\, \delta(\varkappa - \varkappa')$.

From (1.26) and (1.25) we find the necessary and sufficient conditions imposed on the force matrix.

$$\Phi^{qq'}(\varkappa)\, e_{q'}(\varkappa) = \Phi(\varkappa)\, e^{q'}(\varkappa) \tag{2.2}$$

The conditions imposed on $I^{qq'}$ depend on the character of the mass distribution in the cell. In particular, for a continuous distribution of masses with the density $\rho_0 = \text{const}$

$$I^{qq'} e_{q'}(\varkappa) = \rho_0 e^{q'}(\varkappa), \qquad \rho_0 = \frac{1}{a^2}\, e_q(\varkappa)\, I^{qq'}\, \overline{e_{q'}(\varkappa)} \tag{2.3}$$

The spatial dispersion is due here only to the nonlocal character of the elastic interaction.* The transition to the ordinary elastic model is effected upon making the additional assumption of a limited range.

Returning to Equations (1.16), let us consider the transition to long-wave approximations, i.e. to the model of a medium with weak dispersion [1]. In this case the integral operators are replaced by differential operators with constant coefficients. For example, let γ, χ^p, and $\Gamma^{pp'}$ be even functions of k. Then in the second approximation

$$\gamma(k) = \gamma_0 - \gamma_2 k^2, \quad \chi^p(k) = \chi_0^p - \chi_2^p k^2, \quad \Gamma^{pp'}(k) = \Gamma_0^{pp'} - \Gamma_2^{pp'} k^2 \tag{2.4}$$

and the corresponding equations in the (x, t)-representation are of the form $(D = d/dx)$,

$$\begin{aligned} \rho u'' - D^2(\gamma_0 + \gamma_2 D^2)\, u - D(\chi_0^p + \chi_2^p D^2)\, \eta_p &= q \\ I^{pp'}\eta_{p'}'' + D(\chi_0^p + \chi_2^p D^2)\, u + (\Gamma_0^{pp'} + \Gamma_2^{pp'} D^2)\, \eta_{p'} &= \mu^p \end{aligned} \tag{2.5}$$

These equations represent one-dimensional analog of the couple-stress theory of elasticity [4 to 6]. More precisely, in order to obtain one-dimensional equations of the couple-stress theory of elasticity from (2.5), it is necessary to make two assumptions i.e. that (a) there are only two degrees of freedom, i.e. $p = 1$; (b) it is necessary to limit oneself to the zeroth approximation in the first equation, i.e. to set $\gamma_2 = 0$ and $\chi_2 = 0$ in that equation, and to retain the second approximation in the second equation. Whereas the first assumption corresponds to the selection of some specific model, the second assumption is difficult to justify: from the standpoint of the theory of spatial dispersion it does not appear correct.

3. As shown above, the principal interest in many problems is found in the acoustic range of vibrations. As in the case of crystals, the optical frequencies are on the order of 10^{13} sec^{-1}, exceeding markedly the frequencies of mechanical oscillations, in such cases system (1.16) can be simplified considerably and reduced to one equation in the coordinate of the cell center of mass.

Let us first consider the equations of the zeroth approximation,

$$\begin{aligned} \omega^2 \rho\, u(k) - k^2 \gamma_0 u(k) - ik\chi_0^p \eta_p(k) &= -q(k) \\ \omega^2 I^{pp'} \eta_{p'}(k) + ik\chi_0^p u(k) - \Gamma_0^{pp'} \eta_{p'}(k) &= -\mu^p(k) \end{aligned} \tag{3.1}$$

It is easy to see that the equation for the characteristic frequencies for $k = 0$ splits into the equation $\omega = 0$ for the acoustic frequency and the equation $|\Gamma_0^{pp'} - \omega^2 I^{pp'}| = 0$

* In the general case the spatial dispersion is a consequence of both the long range of action and the periodicity (discreteness) of the structure.

for optical frequencies. Stability considerations imply that the optical frequencies cannot vanish. For example, in the case of a diatomic chain this would mean that it breaks down into two simple unrelated chains. Thus, in the acoustic range there exists a matrix $A^\circ(\omega) = \|\Gamma_0 - \omega^2 I\|^{-1}$. But this also implies the existence of the Hermitian matrix

$$A(k, \omega) = \|\Gamma(k) - \omega^2 I\|^{-1} \tag{3.2}$$

in some finite neighborhood of the origin of coordinates of the plane k, ω. This neighborhood is determined by the boundary curve given by the equation $\Gamma(k) - \omega^2 I| = 0$. In the general case the investigation of the boundary curve poses certain difficulties; one can only say that for $k = 0$ it touches the lower optical curve and that $A(k, \omega)$ definitely exists in the long-wave portion of the acoustic range. But for a diatomic chain with interacting proximate neighbors it can be shown that the boundary curve (the broken line in the figure) does not intersect the acoustic branch anywhere, so that $A(k, \omega)$ exists throughout the acoustic range. The range of existence of $A(k, \omega)$ will be referred to as the permissible acoustic range.

The matrix $A(k, \omega)$ can be expressed in terms of $A^\circ(\omega)$ and the coefficients Γ_s of the expansion of the matrix $\Gamma(k)$,

$$\Gamma(k) = \sum_{s=0}^{\infty} (-ik)^s \Gamma_s \tag{3.3}$$

Let $j(m) = \{j_1 j_2 \ldots\}$ be an arbitrary combination of integers j_ν such that

$$1 \leqslant j_\nu \leqslant m, \quad \sum_\nu j_\nu = m$$

By $A^{j(m)}(\omega)$ we denote the bordered product

$$A^{j(m)} = A^0 \ldots A^0 \Gamma_{j_\nu} A^0 \ldots A^0, \qquad j_\nu \in j(m) \tag{3.4}$$

A representation in the form of the series

$$A(k, \omega) = \sum_{m=0}^{\infty} (-ik)^m A^m(\omega), \qquad A^m(\omega) = \sum_j A^{j(m)}(\omega) \tag{3.5}$$

is then valid in the corresponding circle of convergence for $A(k, \omega)$.

We note that in the static case ($\omega = 0$) the matrix $A(k)$ can be expressed in terms of Γ_s and the numerical matrix Γ_0^{-1}.

Using the matrix $A(k, \omega)$, we can solve the second equation of system (1.16) for $\eta_p(k)$ in the permissible acoustic range and eliminate $\eta_p(k)$ from the first equation. The final result was the form

$$\begin{aligned} \omega^2 \rho u(k) - k^2 c(k, \omega)\, u(k) &= -Q(k) \\ \eta_p(k) &= ika_p(k, \omega)\, u(k) + A_{pp'}(k, \omega)\, \mu^{p'}(k) \end{aligned} \tag{3.6}$$

Here we employ the notation

$$\begin{aligned} c(k, \omega) &= \gamma(k) - \overline{\chi^p(k)}\, A_{pp'}(k, \omega)\, \chi^{p'}(k), \qquad a_p(k, \omega) = A_{pp'}(k, \omega)\, \chi^{p'}(k) \\ Q(k) &= q(k) + ik\mu(k), \qquad \mu(k) = -\overline{a_p(k, \omega)}\, \mu^p(k) \end{aligned} \tag{3.7}$$

Equations (3.6) in the permissible acoustic range are the exact equivalent of system

(1.16). In the zeroth approximation in the neighborhood of the acoustic branch ($\omega \sim k$) they assume the form

$$\omega^2 \rho u(k) - c_0 k^2 u(k) = -Q(k)$$
$$\eta_p(k) = i a_p^0(0)\, k u(k) + A_{pp'}^0(0)\, \mu^{p'}(k) \tag{3.8}$$

As expected, the first equation coincides with the equation of one-dimensional homogeneous continuum characterized by the elastic constant c_0. Precisely this elastic constant is determined from macroscopic experiments. Its relationship with the micro-parameters follows from (3.7),

$$c_0 = \gamma_0 - \chi_0^p A_{pp'}^0(0)\, \chi_0^{p'} \tag{3.9}$$

which agrees with the familiar expression [2].

The right side of the equation includes the equivalent external force density Q, which consists of two terms in accordance with (3.7). The first of these is the average external force density q and the second the negative derivative of the micromoment density μ in exact accord with ordinary macroscopic theory. Thus, the average micromoment density μ corresponds to the macroscopic moment density.

The density Q and, in the general case, Equations (3.6) can be interpreted in a similar manner.

Extending our analogy with the macroscopic theory, we naturally interpret the quantity

$$\sigma(k) = c(k, \omega)\, \varepsilon(k) \qquad (\varepsilon(k) = -iku(k)) \tag{3.10}$$

as the stress. The first equation of (3.6) in the (x, t)-representation now becomes

$$\rho u''(x) - D\sigma(x) = Q(x) \tag{3.11}$$

and relation (3.10) corresponds to the operator Hooke's law.

Transforming (1.18) and taking account of (3.6), we find the expression for the elastic energy density,

$$\varphi(x) = \tfrac{1}{2} \sigma(x)\, \varepsilon(x) \tag{3.12}$$

Hence it follows that the stress σ can also be defined as the generalized force corresponding to the generalized displacement – the strain ε, in full accord with ordinary elasticity theory.

Thus, the above representation is more in keeping with the spirit of the macroscopic theory and the mechanics of continuous media than is the representation considered in sec. 1. In particular, it hardly makes sense to introduce a (generally infinite) array of stress tensors corresponding to Equations (1.16). It must be pointed out, however, that whereas Equations (1.16) involve only spatial dispersion, Equation (3.6) encompasses both spatial and temporal dispersion, although the latter does not result in energy dissipation in the present case.

With allowance for (1.15) and (3.2), it follows from (3.7) that $c(k, \omega)$ is an even function of k and ω. For this reason, in the case of weak dispersion Hooke's law in the second approximation can be written as

$$\sigma(x) = (c_0 + c_2 D^2 + c_2' D_t^2)\, \varepsilon(x) \qquad (D_t = d/dt) \tag{3.13}$$

Here c_2 and c_2' are the corresponding coefficients of the expansion of $c(k, \omega)$ in the long-wave neighborhood of the acoustical branch ($\omega \sim k$). Equations (3.11) become

$$J\ddot{u}(x) - (c_0 + c_2 D^2) D^2 u(x) = Q(x) \tag{3.14}$$

where $J = \rho - c_2' D^2$ can, if one wishes, be interpreted as the operator inertial characteristic of the medium.

In conclusion we note that a similar transformation of Equations (1.16) is also possible in the neighborhood of the optical branch.

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